

# Five-dimensional Black Hole and Particle Solution with Non-Abelian Gauge Field

Naoya Okuyama<sup>1</sup> and Kei-ichi Maeda<sup>1,2,3</sup>,

<sup>1</sup> *Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*

<sup>2</sup> *Advanced Research Institute for Science and Engineering, Waseda University, Shinjuku, Tokyo 169-8555, Japan*

<sup>3</sup> *Waseda Institute for Astrophysics, Waseda University, Shinjuku, Tokyo 169-8555, Japan*

(February 7, 2008)

We study the 5-dimensional Einstein-Yang-Mills system with a cosmological constant. Assuming a spherically symmetric spacetime, we find a new analytic black hole solution, which approaches asymptotically “quasi-Minkowski”, “quasi anti-de Sitter”, or “quasi de Sitter” spacetime depending on the sign of a cosmological constant. Since there is no singularity except for the origin which is covered by an event horizon, we regard it as a localized object. This solution corresponds to a magnetically charged black hole. We also present a singularity-free particle-like solution and a non-trivial black hole solution numerically. Those solutions correspond to the Bartnik-McKinnon solution and a colored black hole with a cosmological constant in the 4-dimensions. We analyze their asymptotic behaviors, spacetime structures and thermodynamical properties. We show that there is a set of stable solutions if a cosmological constant is negative.

## I. INTRODUCTION

Recent progress in a superstring theory shows that different string theories are connected with each other via dualities, making them to be unified to the M theory in 11 dimensions [1]. This provides us a motivation to study a higher dimensional gravitational theory. String theory also predicts a boundary layer, a *brane*, on which edges of open strings stand [2]. This suggests a new perspective in cosmology, that is, we are living in a brane world, which is a three-dimensional hypersurface in a higher-dimensional spacetime. In contrast to the already familiar Kaluza-Klein picture in which we live in 4-dimensional spacetime with  $n$  dimensional compactified “internal space”, our world view appears to be changed completely. Particles in the standard model are expected to be confined to the brane, whereas the gravitons propagate in the entire bulk spacetime.

In the brane world cosmological scenario [3], a higher-dimensional black hole solution plays an important role. Our universe is just a domain wall expanding in the black hole background spacetime [4]. The black hole mass gives a contribution to dark radiation through its tidal force. Hence, a higher-dimensional black hole or a globally regular solution with a cosmological constant is now a very interesting subject. In particular, in the context of the AdS/CFT correspondence [5] or proposed dS/CFT correspondence [6], since the 5-dimensional Einstein gravity with a cosmological constant gives a description of 4-dimensional conformal field theory in large  $N$  limit, many authors study such localized objects in 5-dimensions [7].

However, from a view point of brane cosmology, a black hole solution has a singularity in a bulk spacetime, although it is covered by a horizon. If a string theory or M theory is fundamental, such a singularity should not exist. Then, if we can construct some non-singular object in the bulk spacetime, it might be a manifestation of singularity avoidance immanent in a fundamental theory. In 4-dimensions, Bartnik and McKinnon found a

particle-like solution as a globally regular spacetime in a spherically symmetric Einstein-Yang-Mills system with  $SU(2)$  gauge group [8]. Soon after, a colored black hole solution with a nontrivial non-Abelian structure was also found [9]. These solutions were also extended to those in the system with a cosmological constant [10,11,12]. From stability analysis, it turns out that the solution with zero or positive cosmological constant is unstable [13], while those with negative cosmological constant is stable [12,14]. Since a negative cosmological constant is naturally expected in a brane world scenario just as the Randall-Sundrum model [15], the above fact is very interesting. In this paper, then, we study a nontrivial particle-like solution or black hole solution in 5-dimensions with a cosmological constant.

As for non-Abelian gauge field in a bulk spacetime, although gauge interactions are confined on a brane and Yang-Mills fields are expected to exist only in the brane, if our 5-dimensional spacetime is obtained as an effective theory, it may not be the case. In fact, Lukas, et al. [16] showed that  $U(1)$ -field appears in the effective 5-dimensional bulk spacetime, from dimensional reduction of the Hořava-Witten model [1]. We may find non-Abelian gauge field from some other type dimensional reduction of a unified theory.

There is another interesting point to discuss non-Abelian gauge fields in a bulk. Using a brane structure, new mechanism of spontaneous symmetry breaking of gauge interactions have been proposed [17]. In this picture, the present standard model ( $SU(3) \times SU(2) \times U(1)$ ) is obtained on the brane assuming some higher-symmetric gauge interactions such as  $SU(5)$  in the bulk.

Therefore, in this paper, we assume that non-Abelian gauge field appears in 5-dimensional bulk spacetime. In §.II, we first derive the basic equations of a spherically symmetric Einstein-Yang-Mills system in 5-dimensions. With a spherically symmetric ansatz, the gauge potential of  $SU(2)$  Yang-Mills field is decomposed into the

“electric” and “magnetic” part, which derivation is given in Appendix A. There is a nontrivial analytic solution in the case with “magnetic” field, which corresponds to a magnetically charged black hole in 4-dimensions. This analytic solution and its property are examined in §.III. We also present non-trivial particle-like and black hole solutions, which correspond to the Bartnik-McKinnon type and colored black hole type solutions in 4-dimensions, in §.IV. We also analyze those stability in §.V. Summary and discussion follow in §.VI.

## II. BASIC EQUATION

In order to find a black hole and particle-like solution of the 5-dimensional Einstein-Yang-Mills system, we first write down the basic equations. The action is given by

$$S = \frac{1}{16\pi} \int d^5x \sqrt{-g_5} \left[ \frac{1}{G_5} (R - 2\Lambda) - \frac{1}{g^2} \text{Tr} \mathbf{F}^2 \right], \quad (2.1)$$

where  $G_5$  is a 5-dimensional gravitational constant,  $\Lambda$  is a 5-dimensional cosmological constant, and  $g$  is a gauge coupling constant. Now we adopt gauge group is  $\text{SU}(2)$ .  $\mathbf{F} = F_{\mu\nu} dx^\mu \wedge dx^\nu$  is a field strength of the gauge field, which is described by the vector potential  $\mathbf{A} = A_\mu dx^\mu$  as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \quad (2.2)$$

Defining the 5-dimensional Planck mass by  $m_5 = G_5^{-1/3}$  and a fundamental mass scale of the gauge field by  $m_g = g^{-2}$ , we introduce a typical length scale of the present system, which is given by

$$\lambda = \left( \frac{m_g}{m_5^3} \right)^{\frac{1}{2}} = \left( \frac{G_5}{g^2} \right)^{\frac{1}{2}}. \quad (2.3)$$

We will normalize a scale length by this  $\lambda$ .

We consider a spherically symmetric 5-dimensional spacetime, which metric is given by

$$ds^2 = \lambda^2 \left[ -f(t, r) e^{-2\delta(t, r)} dt^2 + \frac{dr^2}{f(t, r)} + r^2 d\Omega_3^2 \right], \quad (2.4)$$

where

$$f(t, r) = 1 - \frac{\mu(t, r)}{r^2} + \epsilon \frac{r^2}{\ell^2}, \quad (2.5)$$

$$d\Omega_3^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (2.6)$$

where we set  $\Lambda = -6\epsilon/(\lambda\ell)^2$  with  $\epsilon = 0$  or  $\pm 1$ , corresponding to the signature of  $\Lambda$ , i.e.  $\epsilon = 1, 0$ , and  $-1$  corresponds to  $\Lambda < 0, \Lambda = 0$ , and  $\Lambda > 0$ , respectively. Note that  $t, r$ , and  $\mu$  are all dimensionless variables. We shall call  $\mu$  a ‘mass’ function.  $\ell$  denotes the ratio of the length scale of a cosmological constant to  $\lambda$ .

From Appendix A, we find a generic form of a spherically symmetric  $\text{SU}(2)$  gauge potential. If we take only

an “electric” part of the field, the gauge potential is given by Eq. (A19), which yields the basic equations as

$$\mu' = \frac{2}{3} r^3 (A' e^\delta)^2, \quad (2.7)$$

$$\dot{\mu} = 0, \quad (2.8)$$

$$\delta' = 0, \quad (2.9)$$

$$\left[ (A' e^\delta)^2 \right]' = 0, \quad (2.10)$$

$$\left[ (A' e^\delta)^2 \right]' + \frac{6}{r} (A' e^\delta)^2 = 0, \quad (2.11)$$

where a dash and dot denote the partial derivative with respect to  $r$  and  $t$  respectively. This equation gives the Reissner-Nordstrom type solution such as

$$\mu = \mathcal{M} - \frac{2Q^2}{3r^2}, \quad (2.12)$$

$$\delta = 0, \quad (2.13)$$

$$A = -\frac{Q}{r^2}. \quad (2.14)$$

This result is same to the case of 4-dimensions.

If the “magnetic” part of the gauge field, which is given by Eq. (A22), appears, we find another basic equations as follows. Using the gauge freedom, we set  $X = 0$ , resulting in the gauge potentials as

$$A_t^a = 0, \quad A_r^a = 0, \quad (2.15)$$

$$A_\psi^a = (0, 0, w), \quad (2.16)$$

$$A_\theta^a = (w \sin \psi, -\cos \psi, 0), \quad (2.17)$$

$$A_\varphi^a = (\cos \psi \sin \theta, w \sin \psi \sin \theta, -\cos \theta), \quad (2.18)$$

where we set  $\phi = w(r, t)$ . With the above ansatz, we find the Einstein equations and Yang-Mills equation of the present system as

$$\mu' = 2r \left[ f w'^2 + f^{-1} e^{2\delta} \dot{w}^2 + \frac{(1 - w^2)^2}{r^2} \right], \quad (2.19)$$

$$\dot{\mu} = 4r f w' \dot{w}, \quad (2.20)$$

$$\delta' = -\frac{2}{r} [w'^2 + f^{-2} e^{2\delta} \dot{w}^2], \quad (2.21)$$

and

$$\frac{1}{r} (r f e^{-\delta} w')' + \frac{2}{r^2} e^{-\delta} w (1 - w^2) = (f^{-1} e^\delta \dot{w})'. \quad (2.22)$$

Eqs. (2.19)-(2.22) look very similar to those in the case of the 4-dimensional Einstein-Yang-Mills system. However, a little difference of the power exponent of  $r$  brings a big difference in the behavior of solutions, as we will see later.

## III. ANALYTIC SOLUTIONS

Now we look for a “magnetic” type static solution of the system (2.19)-(2.22). Dropping the time derivative terms, we find the basic equations as

$$\mu' = 2r \left[ fw'^2 + \frac{(1-w^2)^2}{r^2} \right], \quad (3.1)$$

$$\delta' = -\frac{2}{r}w'^2, \quad (3.2)$$

$$\frac{1}{r}(rfe^{-\delta}w')' + \frac{2}{r^2}e^{-\delta}w(1-w^2) = 0. \quad (3.3)$$

The above differential equations (3.1)-(3.3) have two analytic solutions. One analytic solution is

$$w = \pm 1, \quad \mu = \mathcal{M}, \quad \delta = 0, \quad (3.4)$$

which corresponds to the Schwarzschild or the Schwarzschild-anti de Sitter (or de Sitter) spacetime, which properties are well known.

Another analytic solution is given by

$$w = 0, \quad \mu = \mathcal{M} + 2 \ln r, \quad \delta = 0. \quad (3.5)$$

This solution has a nontrivial geometry. In the 4-dimensional spacetime, this type of solution describes the Reissner-Nordstrom type geometry with a magnetic charge. In the 5-dimensional spacetime,  $2 \ln r$  term appear in the mass function  $\mu$ . Although  $\mu$  diverges, the metric itself approaches that of well-known symmetric spacetime for each  $\epsilon$ , i.e. the Minkowski, de Sitter and anti de Sitter one. We first study the properties of this solution in the following subsections.

### A. Asymptotic structure

Since the mass function diverges, we have to analyze carefully those asymptotic behaviors. For the case of  $\epsilon = 0$ , the Riemann curvature is finite except at  $r = 0$  and vanishes at infinity as

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \rightarrow 288 \left( \frac{\ln r}{r^4} \right)^2. \quad (3.6)$$

For the case of  $\epsilon = \pm 1$ , the Riemann curvature is also finite everywhere except at  $r = 0$  and converges as

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \rightarrow \frac{40}{\ell^4} \quad (3.7)$$

as  $r \rightarrow \infty$ . This finite value just comes from the Ricci curvature. The metric form approaches

$$f(r) \rightarrow 1 + \epsilon \frac{r^2}{\ell^2} \quad (3.8)$$

as  $r \rightarrow \infty$ . These spherically symmetric and static spacetimes are singular only at  $r = 0$ , and seem to approach a “maximally symmetric spacetime”. Therefore we may recognize it as an localized object in such a “maximally symmetric spacetime”.

However we have to analyze those asymptotic behaviors more carefully. The asymptotically flatness condition is mathematically defined using the conformal transformation. We can also extend this formulation for an

asymptotically de Sitter (or anti-de Sitter) spacetime as well as for a higher-dimensional spacetime.

In an asymptotically flat spacetime, we can naturally define a mass of an isolated object, which is called the ADM mass [18]. It is defined by

$$G_5 M_{\text{ADM}} = \frac{1}{16\pi} \oint_{I_0} dS_i [\partial_j h^{ij} - \eta^{ij} \partial_j h_k^k] \quad (3.9)$$

in 5-dimensional spacetime, where  $\eta_{\mu\nu}$  is Minkowski metric and  $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ .  $dS_i$  is an infinitesimal surface element of spacelike infinity  $I_0$ . For the present non-trivial solution with  $\epsilon = 0$ , we find

$$G_5 M_{\text{ADM}} = \lim_{r \rightarrow \infty} \frac{3\pi}{8} \lambda^2 (\mathcal{M} + 2 \ln r) \quad (3.10)$$

which diverges as  $\ln r$ . The coefficient  $3\pi/8$  appears just because Eq. (2.19) yields

$$\mu = \frac{3\pi}{8} \int dv [-T_0^0]. \quad (3.11)$$

For  $\epsilon = -1$ , if the spacetime is asymptotically de Sitter, we can also introduce a conserved mass, which is called the Abbott-Deser mass defined by [19,20]. If the spacetime is asymptotically de Sitter,  $M_{\text{AD}} = M_{\text{ADM}}$ , which diverges again as  $\ln r$ .

In a 5-dimensional asymptotically anti-de Sitter spacetime, we can also define a conserved mass associated with a timelike Killing vector  $\vec{\xi}$  at the 3-sphere  $C$  on conformal infinity  $\mathcal{I}$  as [21,22]

$$G_5 M_\xi[C] := -\frac{\lambda^2 \ell}{16\pi} \oint_C \mathcal{E}_{\mu\nu} \xi^\mu dS^\nu, \quad (3.12)$$

where  $\mathcal{E}_{\mu\nu}$  is the electric part of Weyl tensor defined by

$$\mathcal{E}_{\mu\nu} := \frac{\ell^2}{\Omega^2} C_{\mu\rho\nu\sigma} n^\rho n^\sigma. \quad (3.13)$$

$\Omega$  is a conformal factor and  $n_\mu = \nabla_\mu \Omega$ . In the case of Schwarzschild-anti de Sitter spacetime (3.4), this mass gives  $\mathcal{M}$ . In the non-trivial solution, however, this quantity is calculated on the 3-sphere  $C$  with a radius  $r$  as

$$G_5 M_\xi[C] = \frac{3\pi}{8} \lambda^2 \left[ \mathcal{M} + 2 \ln r - \frac{7}{6} \right]. \quad (3.14)$$

It diverges as  $\ln r$  as  $r \rightarrow \infty$ .

In any case, the “mass” is not finite, which means that “total energy” of the system is not finite. Therefore, strictly speaking, we should not regard it as an isolated object. However, there is no singularity except at  $r = 0$  and the metric form itself approaches either Minkowski or de-Sitter (anti-de Sitter) one. Hence, we call it a “quasi-isolated” object. We remind that we know a similar “isolated” object, i.e. a 4-dimensional self-gravitating global monopole. Its metric is described as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2, \quad (3.15)$$

where  $f(r) \equiv 1 - 2m(r)/r \sim 1 - \alpha - 2M/r + O(1/r^2)$ . In this case, the mass function  $m(r)$  diverges as  $M + \alpha r/2 + O(1/r)$  as  $r \rightarrow \infty$ . In fact the ADM mass diverges. Rescaling the time and radial coordinates as  $r \rightarrow (1 - \alpha)^{1/2}r$  and  $t \rightarrow (1 - \alpha)^{-1/2}t$ , we can rewrite the metric form as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + (1 - \alpha)r^2 d\Omega^2, \quad (3.16)$$

where  $f(r) = 1 - 2\tilde{M}/r$  with  $\tilde{M} = M(1 - \alpha)^{-3/2}$ . This spacetime looks asymptotically flat but has a deficit angle  $\alpha$ . Nucamendi and Sudarsky showed that this spacetime is asymptotically simple but not asymptotically empty [23]. They called it a “quasi-asymptotically flat” spacetime and defined a new mass for a spacetime with a deficit angle, which is a generalization of the ADM mass, using the first law of black hole thermodynamics.

In our case, the mass function diverges as  $\ln r$ , which is less divergent than the case with a deficit angle ( $r^2$  in 5-dimensions). Then we can also call such a spacetime “quasi-asymptotically” flat or “quasi-asymptotically” de Sitter (anti-de Sitter) one.

## B. Spacetime structure: horizon and singularity

This solution has a horizon, where

$$f(r) = 1 - \frac{\mathcal{M} + 2 \ln r}{r^2} + \epsilon \frac{r^2}{\ell^2} = 0. \quad (3.17)$$

We study those horizons and the singularity separately for each value of  $\epsilon$ .

### 1. $\epsilon = 0$

In this case, if  $\mathcal{M} > 1$ , Eq. (3.17) has two roots  $r = r_{\pm} (r_- < r_+)$ , which correspond to two horizons,  $r_+$  corresponds to an event horizon, while  $r_-$  is an inner horizon. A timelike singularity appears at  $r = 0$ . For the case of  $\mathcal{M} = 1$ , two horizons degenerate and a black hole becomes extreme. If  $\mathcal{M} < 1$  there is no horizon, so a naked singularity appears.

### 2. $\epsilon = 1$

This case also has two horizons  $r = r_{\pm} (r_- < r_+)$  if  $\mathcal{M} > \mathcal{M}_{\text{cr}}$ .  $r_+$  and  $r_-$  are an event horizon and an inner horizon, respectively. The critical mass parameter  $\mathcal{M}_{\text{cr}}$  is given by the horizon radius of the extreme case ( $r_{+\text{cr}}$ ), i.e.

$$\mathcal{M}_{\text{cr}} = \frac{1}{2} (1 + r_{+\text{cr}}^2) - 2 \ln r_{+\text{cr}}, \quad (3.18)$$

where

$$r_{+\text{cr}} = \frac{\ell}{2} \left( -1 + \sqrt{1 + \frac{8}{\ell^2}} \right)^{\frac{1}{2}}. \quad (3.19)$$

$\mathcal{M}_{\text{cr}}$  is always larger than unity and it approaches 1 as  $\ell \rightarrow \infty$ , which corresponds to the case of  $\epsilon = 0$ . A timelike singularity appears at  $r = 0$ . For the case of  $\mathcal{M} = \mathcal{M}_{\text{cr}}$ , the black hole is extreme, and for  $\mathcal{M} < \mathcal{M}_{\text{cr}}$ , a horizon disappears.

### 3. $\epsilon = -1$

If a cosmological constant is positive, we expect a cosmological horizon just as a de Sitter spacetime. In fact, we always find at least one horizon. If  $\ell > 2\sqrt{2}$ , and

$$\mathcal{M}_{\text{min}} < \mathcal{M} < \mathcal{M}_{\text{max}}, \quad (3.20)$$

where  $\mathcal{M}_{\text{min}} = g(r_{+\text{cr}})$  and  $\mathcal{M}_{\text{max}} = g(r_{-\text{cr}})$  with  $g(r) = r^2 - 2 \ln r - r^4/\ell^2$  and

$$r_{\pm\text{cr}} = \frac{\ell}{2} \left( 1 \pm \sqrt{1 - \frac{8}{\ell^2}} \right)^{\frac{1}{2}}, \quad (3.21)$$

we find three horizons,  $r_- (< r_{-\text{cr}}) < r_+ (< r_{+\text{cr}}) < r_c$ .  $r_-$ ,  $r_+$  and  $r_c$  are an inner, event, and cosmological horizon, respectively. When  $\mathcal{M} = \mathcal{M}_{\text{max}}$ , the inner and event horizons degenerate ( $r_- = r_+$ ), while if  $\mathcal{M} = \mathcal{M}_{\text{min}}$ , the event and cosmological horizons coincide ( $r_+ = r_c$ ). In the limit of  $\ell \rightarrow 2\sqrt{2}$ ,  $\mathcal{M}_{\text{min}} = \mathcal{M}_{\text{max}} = \mathcal{M}_{\text{cr}} = 3/2 - \ln 2 \approx 0.80685$ , and then three horizons degenerate for  $\mathcal{M} = \mathcal{M}_{\text{cr}}$ .

For other cases, we have only one horizon. The singularity at  $r = 0$  becomes naked.

We summarize the type of horizons in Table 1.

$\epsilon = 0$	$\mathcal{M} < 1$	0
	$\mathcal{M} = 1$	D
	$\mathcal{M} > 1$	I, E
$\epsilon = 1$	$\mathcal{M} < \mathcal{M}_{\text{cr}}$	0
	$\mathcal{M} = \mathcal{M}_{\text{cr}}$	D
	$\mathcal{M}_{\text{cr}} < \mathcal{M}$	I, E
$\epsilon = -1, \ell > 2\sqrt{2}$	$\mathcal{M} < \mathcal{M}_{\text{min}}$	I
	$\mathcal{M} = \mathcal{M}_{\text{min}}$	I, D
	$\mathcal{M}_{\text{min}} < \mathcal{M} < \mathcal{M}_{\text{max}}$	I, E, C
	$\mathcal{M} = \mathcal{M}_{\text{max}}$	D, C
	$\mathcal{M} > \mathcal{M}_{\text{max}}$	C
$\epsilon = -1, \ell \leq 2\sqrt{2}$		C

Table 1 : Type of horizons. I, E, C and D denote an inner, event, cosmological and degenerated horizon, respectively. “0” means no horizon.  $\mathcal{M}_{\text{cr}}$ ,  $\mathcal{M}_{\text{min}}$ , and  $\mathcal{M}_{\text{max}}$  are defined in the text.

### C. Thermodynamical properties

Next we shall see thermodynamical properties. The Hawking temperature is easily calculated from a regularity condition at the event horizon [24]. We find

$$T_{BH} = \frac{1}{2\pi r_+} \left[ 1 - \frac{1}{r_+^2} + 2\epsilon \frac{r_+^2}{\ell^2} \right]. \quad (3.22)$$

The entropy  $S = A/4$  is given by

$$S = \frac{1}{2} \pi^2 r_+^3, \quad (3.23)$$

because the volume of a unit 3-sphere is  $2\pi^2$ . Since the solution does not satisfy the asymptotically flat or de Sitter (or anti de Sitter) conditions, we cannot define gravitational mass. However, if we use the first law of thermodynamics just as the case of a global monopole with a deficit angle [23], we can define thermodynamical mass  $M_T$  as  $dM_T = TdS + \Phi dQ$ . We find

$$M_T = \frac{3\pi}{8} \mathcal{M}, \quad (3.24)$$

where an integration constant is set to be zero. This result shows that the mass parameter  $\mathcal{M}$  essentially denotes the thermodynamical mass.

From Eq. (3.17), the thermodynamical mass is given by the horizon radius as

$$M_T = \frac{3\pi}{8} \left[ r_+^2 \left( 1 + \epsilon \frac{r_+^2}{\ell^2} \right) - 2 \ln r_+ \right]. \quad (3.25)$$

We depict the  $M_T$ - $r_+$  relation in Fig. 1. We find that the horizon radius is smaller than that of the electrically charged Reissner-Nordstrom black hole. We also show the  $M_T$ - $T_{BH}$  relation in Fig. 2. From Eqs. (3.22) and (3.24), we find

$$\frac{dT_{BH}}{dM_T} = -\frac{2}{3\pi^2 r_+^3} \frac{1 - 3/r_+^2 - 2\epsilon r_+^2/\ell^2}{1 - 1/r_+^2 + 2\epsilon r_+^2/\ell^2}, \quad (3.26)$$

which gives a turning point where a specific heat changes its sign. For the case of  $\epsilon = 0$ , a specific heat is positive in  $1 < r_+ < \sqrt{3}$  but becomes negative for  $r_+ > \sqrt{3}$ . (The corresponding critical value for thermodynamical mass is obtained by Eq. (3.25).) For the case of  $\epsilon = 1$ , if  $\ell \leq 2\sqrt{6}$ , a specific heat is always positive. If  $\ell > 2\sqrt{6}$ , a specific heat is positive in  $r_{+cr} < r_+ < r_{-ch}$  and in  $r_+ > r_{+ch}$ , while it is negative in  $r_{-ch} < r_+ < r_{+ch}$ , where

$$r_{\pm ch} = \frac{\ell}{2} \left( 1 \pm \sqrt{1 - \frac{24}{\ell^2}} \right)^{\frac{1}{2}}. \quad (3.27)$$

For the case of  $\epsilon = -1$ , a specific heat is positive in  $r_{-cr} < r_+ < r_{ch}$ , while it is negative for  $r_{ch} < r_+ \leq r_{+cr}$ , where

$$r_{ch} = \frac{\ell}{2} \left( -1 + \sqrt{1 + \frac{24}{\ell^2}} \right)^{\frac{1}{2}}. \quad (3.28)$$

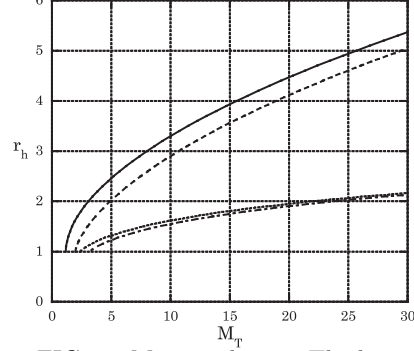


FIG. 1.  $M_T$ - $r_+$  relation. The horizon radius  $r_+$  is depicted in terms of thermodynamical mass  $M_T$  for  $\epsilon = 0$  and  $\epsilon = 1$  by the solid and dotted line respectively. That for the Reissner-Nordstrom solution with same charge for  $\epsilon = 0$  and  $\epsilon = 1$  is given by the dash and dotted-dash line as reference, respectively.

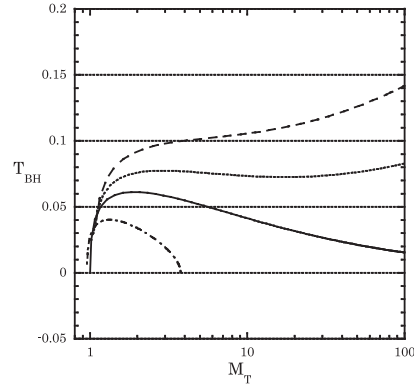


FIG. 2.  $M_T$ - $T_{BH}$  relation. The solid line depicts the relation for  $\epsilon = 0$ , while the dotted and dash lines represent those for  $\epsilon = 1$  with  $\ell = 6.0$  and  $\ell = 4.0$ , and the dotted-dash line correspond to that for  $\epsilon = -1$  with  $\ell = 5.0$  respectively.

## IV. NUMERICAL SOLUTIONS

Just as 4-dimensions [8,9,12,14], we can find non-trivial structure of self-gravitating Yang-Mills field. We obtain those solutions numerically. We discuss two cases; a particle solution and a black hole, separately. Here, we analyze only the case of  $\epsilon = 0$  or 1.

### A. Particle solution

In the case of a particle solution, we have to impose regularity at the origin  $r = 0$ . Since Eqs. (2.19)-(2.22) are invariant under the transformation of  $w \rightarrow -w$ , we can set  $w(0) > 0$  without loss of generality. Expanding

$\mu$  and  $w$  around  $r = 0$ , we find those behaviors near the origin as

$$\mu(r) = 4b^2 r^4 + O(r^5) \quad (4.1)$$

$$\delta(r) = -4b^2 r^2 + \frac{4}{3}b^2 \left( \frac{4\epsilon}{3\ell^2} - 3b - 8b^2 \right) r^4 + O(r^5) \quad (4.2)$$

$$w(r) = 1 + br^2 - \frac{b}{6} \left( \frac{4\epsilon}{3\ell^2} - 3b - 8b^2 \right) r^4 + O(r^5) \quad (4.3)$$

with one free parameter  $b$ . Using this boundary condition, we integrate the basic equations by the Runge-Kutta method.

For the case of  $\epsilon = 0$ , we find the solutions which metrics are regular in whole spacetime and approach to Minkowski as  $r \rightarrow \infty$  for  $b_{\min} < b < 0$ , where  $b_{\min} \approx -0.635607$ . We show the numerical result in Figs. 3 ~ 5.

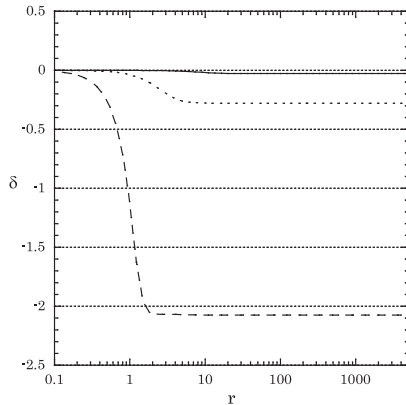


FIG. 3. The metric function  $\delta(r)$  for a particle-like solution with  $\epsilon = 0$ . The solid, dotted and dash lines depict those for  $b = -0.01, -0.1$  and  $-0.5$ , respectively.

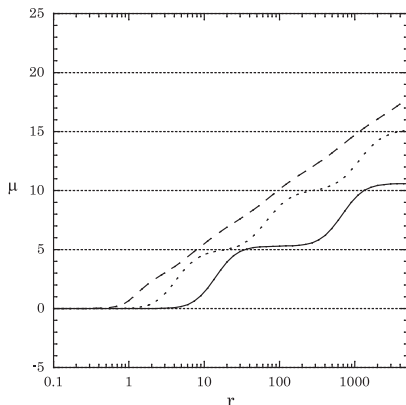


FIG. 4. The mass function  $\mu(r)$  for a particle-like solution with  $\epsilon = 0$ . The solid, dotted and dash lines depict those for  $b = -0.01, -0.1$  and  $-0.5$ , respectively.

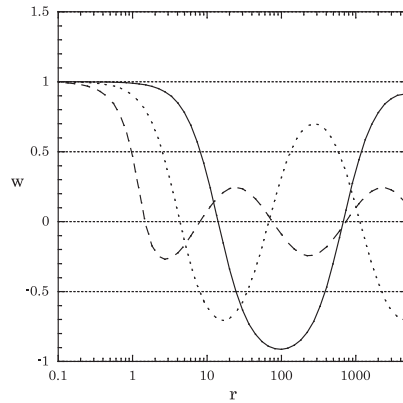


FIG. 5. The potential function  $w(r)$  for a particle-like solution with  $\epsilon = 0$ . The solid, dotted and dash lines depict those for  $b = -0.01, -0.1$  and  $-0.5$ , respectively.

The potential function  $w$  is oscillating between  $\pm 1$  and the mass function  $\mu$  is increasing without bound just as a step function. As we show in Appendix B, there is no finite mass particle-like solution. The mass function increases as  $\ln r$  asymptotically just as the analytic solution (3.5). The period of oscillations of  $w$  is the same as that of steps in  $\mu$  and it is constant in terms of  $\ln r$ . This behaviour is easily understood by solving the basic equations in the asymptotic far region ( $r \rightarrow \infty$ ), which analytic forms are given in Appendix C. We can check that the asymptotic solution is consistent with our numerical solutions. The oscillations of  $w$  and the periodic steps in  $\mu$  are caused by infinite number of instantons (see Appendix C).

For the case of  $\epsilon = 1$ , we also find a regular solution for  $b_{\min} < b < 0$ .  $b_{\min}$  depends on  $\ell$ , and decreases as  $\ell$  decreases. For example,  $b_{\min} \approx -0.644036$  for  $\ell = 10$ ,  $b_{\min} \approx -1.105002$  for  $\ell = 1$ . We show the numerical result in Figs. 6 ~ 8.

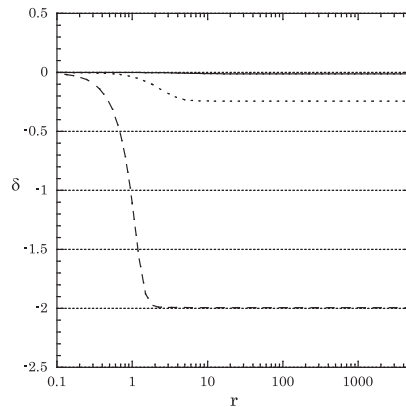


FIG. 6. The metric function  $\delta(r)$  for a particle-like solution with  $\epsilon = 1$ . The solid, dotted and dash lines depict those for  $b = -0.01, -0.1$  and  $-0.5$  respectively. We set  $\ell = 10$ .

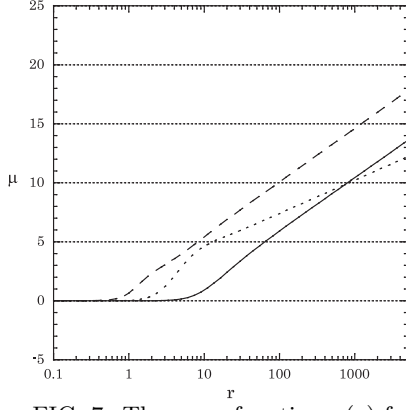


FIG. 7. The mass function  $\mu(r)$  for a particle-like solution with  $\epsilon = 1$ . The solid, dotted and dash lines depict those for  $b = -0.01, -0.1$  and  $-0.5$  respectively. We set  $\ell = 10$ .

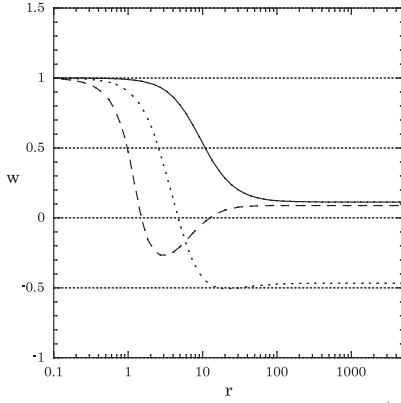


FIG. 8. The potential function  $w(r)$  for a particle-like solution with  $\epsilon = 1$ . The solid, dotted and dash lines depict those for  $b = -0.01, -0.1$  and  $-0.5$  respectively. We set  $\ell = 10$ .

In this case, the potential  $w$  does not oscillate, and converge to some value  $w_\infty$ , then the number of node is finite. The mass function increases monotonically as

$$\mu \rightarrow 2(1 - w_\infty^2)^2 \ln r \quad (4.4)$$

as  $r \rightarrow \infty$ . This behaviour is also understood by solving the asymptotic solution, which is given in Appendix C.

### B. Black hole solution

Next we show a non-trivial black hole solution. To find a black hole solution, we have to impose a boundary condition at a horizon  $r_h$ . The horizon is defined by  $f(r_h) = 0$ , which gives

$$\mu(r_h) = r_h^2 \left( 1 + \epsilon \frac{r_h^2}{\ell^2} \right). \quad (4.5)$$

Here we set  $\delta(r_h) = 0$ . The proper time of the observer at infinity (i.e.  $\delta(\infty) = 0$ ) is obtained by transformation  $t' = e^{-\delta(\infty)} t$ . From Eq. (2.22),  $w'(r_h)$  has to satisfy

$$w'(r_h) = -\frac{w_h(1 - w_h^2)}{r_h [1 + 2\epsilon r_h^2/\ell^2 - (1 - w_h^2)^2/r_h^2]}, \quad (4.6)$$

where  $w_h = w(r_h)$ . There is only one free parameter  $w_h$  for a given value of  $r_h$ . Since Eqs. (2.19)-(2.22) are invariant under the transformation of  $w \rightarrow -w$ , we can set  $w_h > 0$  without loss of generality.

For the solution with  $w_h > 1$ , we find that the curvature diverges at finite distance. Then, we obtain a numerical solution for  $0 \leq w_h \leq 1$  for a given  $r_h$ . We show the results in Figs. 9 ~ 11.

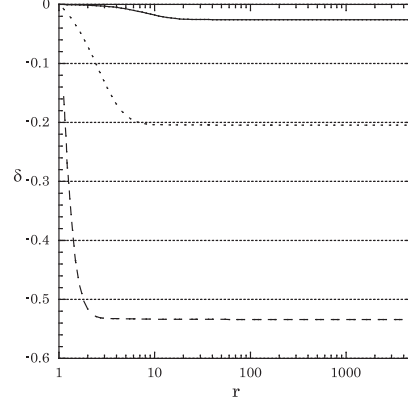


FIG. 9. The metric function  $\delta(r)$  for a black hole solution with  $\epsilon = 0$ . The solid, dotted and dash lines depict that of  $w_h = 0.99, 0.9$  and  $0.5$ , respectively.

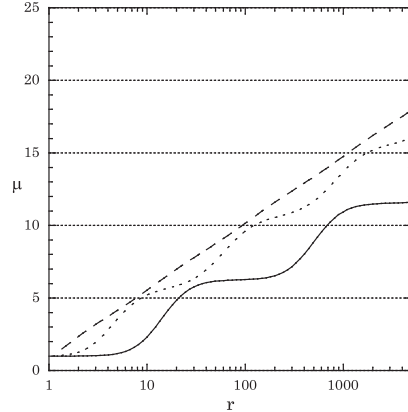


FIG. 10. The mass function  $\mu(r)$  for a black hole solution with  $\epsilon = 0$ . The solid, dotted and dash lines depict that of  $w_h = 0.99, 0.9$  and  $0.5$ , respectively.

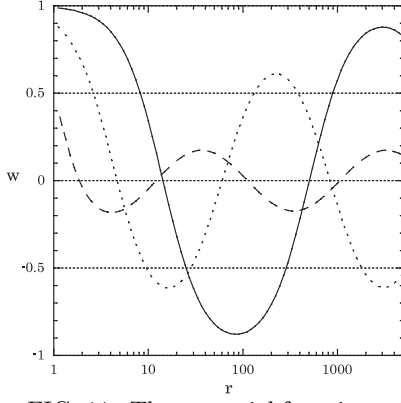


FIG. 11. The potential function  $w(r)$  for a black hole solution with  $\epsilon = 0$ . The solid, dotted and dash lines depict that of  $w_h = 0.99, 0.9$  and  $0.5$ , respectively.

Asymptotic behaviour is similar to that of the particle solution. The potential  $w$  oscillates infinitely with a constant period in terms of  $\ln r$ . For any solutions with  $0 < w_h < 1$ , we find that the mass function  $\mu(r)$  diverges as  $\ln r$  at large distance.

We also show the case with  $\epsilon = 1$  in Figs. 12 ~ 14.

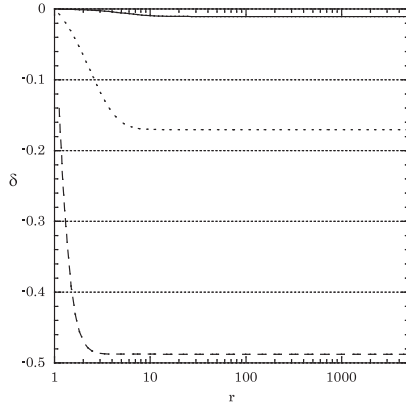


FIG. 12. The metric function  $\delta(r)$  for a black hole solution with  $\epsilon = 1$ . The solid, dotted and dash lines depict that of  $w_h = 0.99, 0.9$  and  $0.5$  respectively. We set  $\ell = 10$ .

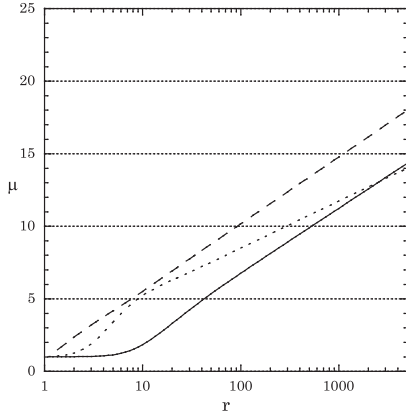


FIG. 13. The mass function  $\mu(r)$  for a black hole solution with  $\epsilon = 1$ . The solid, dotted and dash lines depict that of  $w_h = 0.99, 0.9$  and  $0.5$  respectively. We set  $\ell = 10$ .

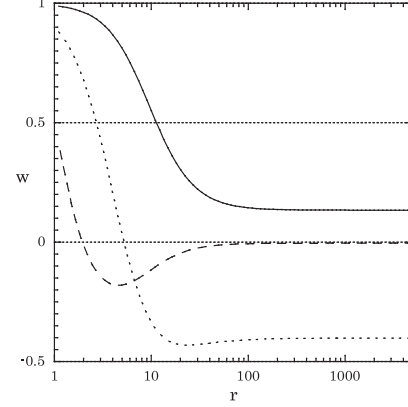


FIG. 14. The potential function  $w(r)$  for a black hole solution with  $\epsilon = 1$ . The solid, dotted and dash lines depict that of  $w_h = 0.99, 0.9$  and  $0.5$  respectively. We set  $\ell = 10$ .

This also shows the similar asymptotic behaviours to a particle solution with  $\epsilon = 1$ .

As for the thermodynamical properties, we find the Hawking temperature as

$$T_{\text{BH}} = \frac{e^{\delta(\infty)}}{2\pi r_h} \left[ 1 - \frac{(1 - w_h^2)^2}{r_h^2} + 2\epsilon \frac{r_h^2}{\ell^2} \right], \quad (4.7)$$

where  $\delta(\infty)$  comes from our coordinate condition, that is, we set  $\delta(r_h) = 0$ . Thermodynamical mass  $M_T$  is found from the first law of black hole thermodynamics,  $d\mathcal{M}_T = TdS + \Phi dQ$ . In order to calculate  $M_T$ , fixing  $w_\infty = w(\infty)$ , we solve a black hole solution because a “global charge” is proportional to  $(1 - w_\infty^2)$ . The result is shown in Fig. 15.

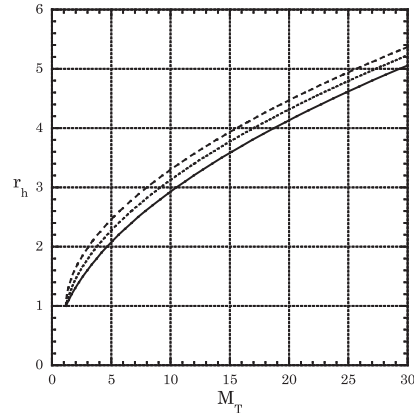


FIG. 15. The  $r_h$ - $\mathcal{M}_T$  relation. The solid, dotted and dash lines depict those for  $w_\infty = 0.9, 0.5$  and  $0.0$  (analytic solution).

We numerically confirm that the thermodynamical mass  $\mathcal{M}_T$  is equal to



$$\mathcal{M}_T = \lim_{r \rightarrow \infty} \frac{3\pi}{8} [\mu - 2(1 - w^2)^2 \ln r]. \quad (4.8)$$

## V. STABILITY

In this section, we analyze stability of the static solutions obtained above. We perturb the metric and potential as

$$\mu(r, t) = \mu_0(r) + \mu_1(r)e^{i\omega t}, \quad (5.1)$$

$$\delta(r, t) = \delta_0(r) + \delta_1(r)e^{i\omega t}, \quad (5.2)$$

$$w(r, t) = w_0(r) + w_1(r)e^{i\omega t}, \quad (5.3)$$

where  $\mu_0(r)$ ,  $\delta_0(r)$ , and  $w_0(r)$  are those of the static solution obtained in previous section. Substituting them into Einstein equations and Yang-Mills equation, we find the perturbation equations as

$$\mu'_1 = 2r \left[ 2f_0 w'_0 w'_1 - \frac{w_0'^2}{r^2} \mu_1 - \frac{4(1 - w_0^2)w_0}{r^2} w_1 \right], \quad (5.4)$$

$$\mu_1 = 4r f_0 w'_0 w_1, \quad (5.5)$$

$$\delta'_1 = -\frac{4}{r} w'_0 w'_1, \quad (5.6)$$

and

$$\begin{aligned} & -\frac{1}{r^3} (r f_0 e^{-\delta_0} w'_0)' f_0^{-1} \mu_1 - f_0 e^{-\delta_0} w'_0 \left( \frac{1}{r^2} f_0^{-1} \mu_1 + \delta_1 \right)' \\ & + \frac{1}{r} (r f_0 e^{-\delta_0} w'_1)' + \frac{2}{r^2} e^{-\delta_0} (1 - 3w_0^2) w_1 = -\omega^2 f_0^{-1} e^{\delta_0} w_1, \end{aligned} \quad (5.7)$$

where  $f_0 = 1 - \mu_0(r)/r^2 + \epsilon r^2/\ell^2$ . Eq. (5.4) is derived from Eq. (5.5) by differentiation.

We introduce a tortoise coordinate  $r_*$  such that

$$\frac{dr_*}{dr} = e^{\delta_0} f_0^{-1} \quad (5.8)$$

and define  $\chi = w_1 r^{\frac{1}{2}}$ . Then, by substituting Eqs. (5.4)-(5.6), Eq. (5.7) turns to be a single uncoupled equation as

$$-\frac{d^2 \chi}{dr_*^2} + V(r_*) \chi = \omega^2 \chi, \quad (5.9)$$

where

$$\begin{aligned} V(r_*) = f_0 e^{-\delta_0} \left\{ \frac{2}{r^2} e^{-\delta_0} (3w_0^2 - 1) + \frac{r^{-\frac{1}{2}}}{2} (r^{-\frac{1}{2}} f_0 e^{-\delta_0})' \right. \\ \left. + \frac{4}{r} [f_0 e^{-\delta_0} w_0'^2]' \right\}. \end{aligned} \quad (5.10)$$

When  $V(r_*)$  is positive definite, we can prove its stability as follows: Multiplying Eq. (5.9) by  $\bar{\chi}$  and integrating

from  $r = r_+(r_* = -\infty)$  in the case of a black hole solution or  $r = 0$  ( $r_* = 0$ ) in the case of a particle solution to  $r = \infty$  ( $r_* = r_{*,\max}(< \infty)$ ), Eq. (5.9) is written as

$$\begin{aligned} & - \left[ \bar{\chi} \frac{d\chi}{dr_*} \right]_{r=r_+ \text{ (or } 0)}^{r=\infty} + \int \left[ \left| \frac{d\chi}{dr_*} \right|^2 + V(r) |\chi|^2 \right] dr_* \\ & = \omega^2 \int |\chi|^2 dr_*. \end{aligned} \quad (5.11)$$

We assume that  $w_1 \rightarrow 0$  at infinity ( $r \rightarrow \infty$  ( $r_* \rightarrow r_{*,\max}$ )). Then,  $\bar{\chi} d\chi/dr_* \rightarrow 0$ . In the case of a black hole,  $\chi$  must be ingoing at horizon ( $r = r_+$  ( $r_* = -\infty$ )). Since the potential  $V$  vanishes at the horizon, the ingoing wave condition gives  $\chi \sim e^{i\omega r_*}$ . If we assume that  $\text{Im}\omega < 0$ , then  $\bar{\chi} d\chi/dr_* \rightarrow 0$  at horizon is obtained. Because  $V(r)$  is positive definite, Eq. (5.11) implies that eigenvalue  $\omega$  is real, that is,  $\text{Im}\omega = 0$ , which contradicts with the above assumption. Hence, we conclude that  $\text{Im}\omega \geq 0$ , which means that the present system is stable. In the case of a particle solution, we should impose  $w_1 = 0$  at the origin ( $r = 0$  ( $r_* = 0$ ))), then we find  $\bar{\chi} d\chi/dr_* \rightarrow 0$ . If  $V(r)$  is positive definite, Eq. (5.11) again implies that the eigenvalue  $\omega$  is real. Hence, in both cases, we obtain that solutions with a positive definite potential  $V(r_*)$  are stable.

For the analytic solution (3.5), we find

$$V(r_*) = \frac{f_0}{4r^4} \left[ 5\mathcal{M} - 4 + 10 \ln r - 9r^2 + 3\epsilon \frac{r^4}{\ell^2} \right]. \quad (5.12)$$

In the case of  $\epsilon = 0$  or  $-1$ ,  $V(r_*)$  is negative at large distance  $r$ . While, in the case of  $\epsilon = 1$ , we see that  $V(r_*)$  is positive definite for sufficiently large  $\mathcal{M}$ , i.e. for

$$\mathcal{M} > \frac{1}{5} \left( 4 - 10 \ln r_p + 9r_p^2 - 3\frac{r_p^4}{\ell^2} \right), \quad (5.13)$$

where  $r_p = (9 - \sqrt{81 - 120/\ell^2})\ell/12$  and  $\ell > \sqrt{40/27}$ .

For the numerical solutions, we also find a positive definite potential  $V(r_*)$  only for the case of  $\epsilon = 1$ . For a particle solution, for example, the positive definite potential is found in the parameter range of  $-0.010368 < b < 0$  for  $\ell = 10$ , and  $-0.654211 < b < 0$  for  $\ell = 1$ . We depict some typical potentials in Figs. 16 and 17.

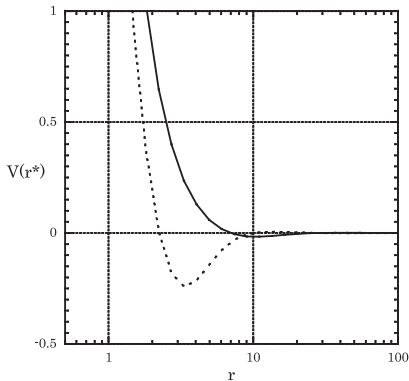


FIG. 16. Potential  $V(r_*)$  for  $\epsilon = 0$ . The solid and dotted line denote those for  $b = -0.01$  and  $b = -0.1$ . There is a negative region ( $V(r_*) < 0$ ) for both potentials.

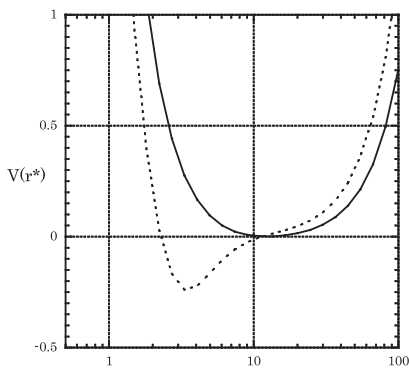


FIG. 17. Potential  $V(r_*)$  for  $\epsilon = 1$ . The solid and dotted line denote those for  $b = -0.01$  and  $b = -0.1$ . There is a negative region ( $V(r_*) < 0$ ) for  $b = -0.1$  but positive definite for  $b = -0.01$ . We set  $\ell = 10$ .

If  $V(r_*)$  is positive definite, we conclude that the system is stable, however, we cannot predict anything if  $V(r_*)$  is not positive definite. We have to solve Eqs. (5.4)-(5.7) as eigenvalue problem numerically. We leave this as a future work and in this paper, we do not discuss it further.

## VI. SUMMARY AND DISCUSSION

In this paper we have studied a spherically symmetric Einstein-SU(2)-Yang-Mills system in 5-dimensions. If we consider only “electric part” of Yang-Mills field, we find the 5-dimensional Reissner-Nordstrom black hole solution. As for “magnetic part” of Yang-Mills field, apart from a trivial Schwarzschild (Schwarzschild-de Sitter, or Schwarzschild-anti de Sitter) solution, we find non-trivial analytic solution, which corresponds to a magnetically charged black hole. (It turns out to be just a Reissner-Nordstrom solution in 4-dimensional case). This non-trivial solution shows that a gravitational “mass” is infinite and the spacetime does not satisfy asymptotically flat, de-Sitter, or anti-de Sitter condition, in contrast to

the case of 4-dimensions. However, its metric approaches either Minkowski or de-Sitter (or anti-de Sitter). We also find that there is no singularity except one at the origin which is covered by a horizon. Hence we call its behaviour at infinity “quasi-asymptotically” flat, de Sitter, or anti-de Sitter and we regard our solution as a localized object. We analyze the spacetime structure and thermodynamical properties. They show that mass parameter  $\mathcal{M}$  in the solution is regarded as a thermodynamical mass, which satisfies the first law of the black hole thermodynamics.

For the case with zero or negative cosmological constant, we also find numerically a particle-like solutions, which has no singularity, and black hole solutions with non-trivial structures of Yang-Mills field. Although, for both cases, the mass function diverges as  $\ln r$ , they satisfy “quasi-asymptotically” flat or anti-de Sitter conditions. If  $\Lambda = 0$ , in contrast to the case of 4-dimension, Yang-Mills field oscillates and has infinite number of nodes. For a negative cosmological constant, Yang-Mills field potential settles to some constant, which is similar to that in 4-dimensional case.

From stability analysis, we find that there is a set of stable solutions if a cosmological constant is negative. This result is very similar to the 4-dimensional case, in which Bartnik-McKinnon solution and a colored black hole is unstable, while those extended to the case with a negative cosmological constant become stable.

Since we find a stable non-singular solution in the 5-dimension, if we apply it to a brane world scenario, we may find some interesting effect on the brane dynamics. We will publish its analysis in a separated paper.

## ACKNOWLEDGMENTS

We would like to thank Takashi Torii for useful discussions and comments. This work was partially supported by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture (Nos. 14047216, 14540281) and by the Waseda University Grant for Special Research Projects.

## APPENDIX A: FIVE-DIMENSIONAL SPHERICALLY SYMMETRIC $SU(2)$ GAUGE FIELD

Here we calculate a generic form of spherically symmetric  $SU(2)$  Yang-Mills field in 5-dimensional spacetime. In the case of four dimensions, Witten gave its generic form [25], which was called the Witten ansatz and proved by Forgács and Manton [26]. Forgács and Manton presented how to find a generic form of spherically symmetric Yang-Mills field in arbitrary dimensions. We just follow their method.

Suppose we have some symmetry of spacetime generated by a vector  $\vec{\eta}$ . A tensor field must be invariant under an infinitesimal transformation generated by  $\vec{\eta}$ , i.e. the Lie derivative of this tensor field with respect to  $\vec{\eta}$  must vanish. However, in the case of a gauge field  $A_\mu$ , there is a gauge freedom, by which we can weaken this condition such that there exists an infinitesimal gauge transformation equivalent to a spacetime transformation, that is

$$\mathcal{L}_{\vec{\eta}} A_\mu = D_\mu W \equiv \partial_\mu W - [A_\mu, W] \quad (\text{A1})$$

for some scalar field  $W$  [27].

Suppose that a  $D$ -dimensional Riemannian manifold  $M$  has some spacetime symmetry represented by  $N$ -dimensional isometry group  $S$ . This isometry group is generated by  $N$  Killing vectors  $\vec{\xi}_{(n)}$  ( $1 \leq n \leq N$ ), which commutation relations is given by

$$[\vec{\xi}_{(m)}, \vec{\xi}_{(n)}]^\mu = f_{mnp} \xi_{(p)}^\mu, \quad (\text{A2})$$

where  $f_{mnp}$  is a structure constant. We assume that the orbit  $X = \{ap \in M | a \in S\}$  for some point  $p \in M$  is  $N'$ -dimensional submanifold of  $M$ . Then we choose the local coordinate system as

$$x^\mu = (x^i, y^\alpha), \quad 1 \leq i \leq D - N', \quad 1 \leq \alpha \leq N' \quad (\text{A3})$$

so that a hypersurface of  $x^i = \text{constant}$  defines the orbit space  $X$ . By Frobenius' theorem, the above Killing vectors are orthogonal to  $X$ , then  $\xi_{(n)}^\mu = (0, \xi_{(n)}^\alpha)$  in this coordinate system.

Because the isometry group is  $N$ -dimensional Lie group, we can define right and left invariant vectors ( $\vec{\xi}_{(n)}^R$  and  $\vec{\xi}_{(n)}^L$ ) as

$$\mathcal{L}_{\vec{\xi}_{(n)}^R} s = s J^{(n)}, \quad \mathcal{L}_{\vec{\xi}_{(n)}^L} s = -J^{(n)} s \quad (\text{A4})$$

for any  $s \in S$ , where  $J^{(n)}$  is the generator of Lie group  $S$  associated by the Killing vector  $\vec{\xi}_{(n)}$ . Then both  $\vec{\xi}_{(n)}^R$  and  $\vec{\xi}_{(n)}^L$  have same commutation relations as those of  $\vec{\xi}_{(n)}$ . We also define those covariant vector fields  $\xi_{(n)\hat{\alpha}}^R, \xi_{(n)\hat{\alpha}}^L$  by

$$\xi_{(m)\hat{\alpha}}^R \xi_{(n)}^{R\hat{\alpha}} = \delta_{mn}, \quad \xi_{(m)\hat{\alpha}}^L \xi_{(n)}^{L\hat{\alpha}} = \delta_{mn}. \quad (\text{A5})$$

For a fixed point  $q \in X$ ,  $R = \{a \in S | aq = q\} \subset S$  is an invariant subgroup of  $S$  with dimension  $N - N'$ , and the quotient group  $S/R$  is diffeomorphic to  $X$ . So we can adopt the same coordinates  $y^\alpha$  in  $X$  for the coset  $Rs \in S/R$ . We take that the other coordinate components are expressed by  $y^\omega$  ( $1 \leq \omega \leq N - N'$ ), which correspond to those of the isotropy group  $R$ . If we fix the origin  $s_0(y^\alpha) \in Rs(y^\alpha)$  for each coset in smooth way, then any element  $s$  of  $S$  is written uniquely with coordinates  $y^{\hat{\alpha}} = (y^\omega, y^\alpha)$ , as

$$s(y^{\hat{\alpha}}) = r(y^\omega) s_0(y^\alpha) \quad (\text{A6})$$

for some  $r \in R$ . In this coordinate, right invariant vector  $\vec{\xi}_{(n)}^R$  is expressed with Killing vector  $\vec{\xi}_{(n)}$  as

$$\xi_{(n)}^{R\hat{\alpha}} = (\xi_{(n)}^{R\omega}, \xi_{(n)}^\alpha). \quad (\text{A7})$$

By the above definition, we find a generic form of gauge potential  $A_\mu(x^\mu)$  with a gauge symmetry  $G$  and a spacetime symmetry  $S$  as

$$A_i(x^i, y^\alpha) = A_i(x^i), \quad A_\alpha(x^i, y^\alpha) = \Phi_n(x^i) \xi_{(n)\alpha}^L(y^\alpha) |_{y^\omega = y_0^\omega}, \quad (\text{A8})$$

where  $A_i(x^i)$  and  $\Phi_n(x^i)$  satisfy the conditions

$$\begin{aligned} f_{mnp} \Phi_p + [\Phi_m, \Phi_n] &= 0 \quad (\forall m, n > N'), \\ \partial_i \Phi_n - [A_i, \Phi_n] &= 0 \quad (\forall i, n > N'), \end{aligned} \quad (\text{A9})$$

and  $y^\omega = y_0^\omega$  is a coordinate component of a unit element of the isotropy group  $R$ . Incidentally,  $W_n$  in Eq. (A1) are obtained as

$$W_n(x^i, y^\alpha) = -\Phi_m(x^i) \xi_{(n)}^{R\omega}(y^\alpha) \xi_{(m)\omega}^L(y^\alpha)|_{y^\omega=y_0^\omega}. \quad (\text{A10})$$

Applying this formalism the 5-dimensional spherically symmetric SU(2) gauge field, we obtain a generic form of the gauge potential  $A_\mu(x^\mu)$ . We assume the isometric group is SO(4). In this coordinate system (2.4), the orbit  $X$  is given as  $t, r = \text{constant}$ , and then  $x^\mu$  is divided into  $x^i = (t, r)$  and  $y^\alpha = (\psi, \theta, \varphi)$ .

The Killing vectors are given as

$$\begin{aligned} \vec{\xi}_{(1)} &= (0, 0, -\cos \theta, \cot \psi \sin \theta, 0), \\ \vec{\xi}_{(2)} &= (0, 0, -\sin \theta \cos \varphi, -\cot \psi \cos \theta \cos \varphi, \frac{\cot \psi \sin \varphi}{\sin \theta}), \\ \vec{\xi}_{(3)} &= (0, 0, -\sin \theta \sin \varphi, -\cot \psi \cos \theta \sin \varphi, -\frac{\cot \psi \cos \varphi}{\sin \theta}), \\ \vec{\xi}_{(4)} &= (0, 0, 0, -\cos \varphi, \cot \theta \sin \varphi), \\ \vec{\xi}_{(5)} &= (0, 0, 0, -\sin \varphi, -\cot \theta \cos \varphi), \\ \vec{\xi}_{(6)} &= (0, 0, 0, 0, -1), \end{aligned} \quad (\text{A11})$$

and the structure constants  $f_{mnp}$  are found to be

$$f_{124} = 1, \quad f_{135} = 1, \quad f_{236} = 1, \quad f_{456} = 1, \quad (\text{A12})$$

with totally antisymmetrized other components.

Next we adopt the local coordinate system which satisfies Eq. (A6) in SO(4). It is given as 4-dimensional Euler angle  $(\alpha, \beta, \chi, \psi, \theta, \varphi)$  as

$$\begin{aligned} s(\alpha, \beta, \chi, \psi, \theta, \varphi) &= r(\alpha, \beta, \chi) s_0(\psi, \theta, \varphi) \\ &= R_{xy}(\alpha) R_{yz}(\beta) R_{xy}(\chi) R_{zu}(\psi) R_{yz}(\theta) R_{xy}(\varphi), \end{aligned} \quad (\text{A13})$$

where  $R_{pq}$  denotes a rotation matrix of the  $pq$ -plane. Note that  $R_{xy}(\alpha) R_{yz}(\beta) R_{xy}(\chi)$  describes any element of an isotropy group  $R$ .

In this coordinate system, the right invariant vector  $\vec{\xi}_n^R$  and the covariant left invariant vector  $\vec{\xi}_n^L$  are

$$\begin{aligned} \xi_{(1)}^{R\hat{\alpha}} &= \left( -\frac{\sin \chi \sin \theta}{\sin \beta \sin \psi}, -\frac{\cos \chi \sin \theta}{\sin \psi}, \frac{\cot \beta \sin \chi \sin \theta}{\sin \psi}, -\cos \theta, \cot \psi \sin \theta, 0 \right), \\ \xi_{(2)}^{R\hat{\alpha}} &= \left( \frac{\cos \chi \sin \varphi + \sin \chi \cos \theta \cos \varphi}{\sin \beta \sin \psi}, -\frac{\sin \chi \sin \varphi - \cos \chi \cos \theta \cos \varphi}{\sin \psi}, \right. \\ &\quad \left. -\frac{\cot \beta \sin \chi \cos \theta \cos \varphi + (\cos \psi \cot \theta + \cot \beta \cos \chi) \sin \varphi}{\sin \psi}, -\sin \theta \cos \varphi, -\cot \psi \cos \theta \cos \varphi, \frac{\cot \psi \sin \varphi}{\sin \theta} \right), \\ \xi_{(3)}^{R\hat{\alpha}} &= \left( -\frac{\cos \chi \cos \varphi - \sin \chi \cos \theta \sin \varphi}{\sin \beta \sin \psi}, \frac{\sin \chi \cos \varphi + \cos \chi \cos \theta \sin \varphi}{\sin \psi}, \right. \\ &\quad \left. -\frac{\cot \beta \sin \chi \cos \theta \sin \varphi - (\cos \psi \cot \theta + \cot \beta \cos \chi) \cos \varphi}{\sin \psi}, -\sin \theta \sin \varphi, -\cot \psi \cos \theta \sin \varphi, -\frac{\cot \psi \cos \varphi}{\sin \theta} \right), \\ \xi_{(4)}^{R\hat{\alpha}} &= \left( 0, 0, -\frac{\sin \varphi}{\sin \theta}, 0, -\cos \varphi, \cot \theta \sin \varphi \right), \\ \xi_{(5)}^{R\hat{\alpha}} &= \left( 0, 0, \frac{\cos \varphi}{\sin \theta}, 0, -\sin \varphi, -\cot \theta \cos \varphi \right), \\ \xi_{(6)}^{R\hat{\alpha}} &= (0, 0, 0, 0, 0, -1), \end{aligned} \quad (\text{A14})$$

and

$$\begin{aligned} \xi_{(1)\hat{\alpha}}^L &= (0, 0, 0, \cos \beta, \sin \beta \cos \chi \sin \psi, \sin \beta \sin \chi \sin \psi \sin \theta), \\ \xi_{(2)\hat{\alpha}}^L &= (0, 0, 0, -\cos \alpha \sin \beta, -\sin \alpha \sin \chi \sin \psi + \cos \alpha \cos \beta \cos \chi \sin \psi, \end{aligned}$$

$$\begin{aligned}
& \sin \alpha \cos \chi \sin \psi \sin \theta + \cos \alpha \cos \beta \sin \chi \sin \psi \sin \theta), \\
\xi_{(3)\hat{\alpha}}^L &= (0, 0, 0, \sin \alpha \sin \beta, -\cos \alpha \sin \chi \sin \psi - \sin \alpha \cos \beta \cos \chi \sin \psi, \\
& \cos \alpha \cos \chi \sin \psi \sin \theta - \sin \alpha \cos \beta \sin \chi \sin \psi \sin \theta), \\
\xi_{(4)\hat{\alpha}}^L &= (0, \cos \alpha, \sin \alpha \sin \beta, 0, \cos \alpha \cos \chi \cos \psi - \sin \alpha \cos \beta \sin \chi \cos \psi, \\
& \sin \alpha \sin \beta \cos \theta + \cos \alpha \sin \chi \cos \psi \sin \theta + \sin \alpha \cos \beta \cos \chi \cos \psi \sin \theta), \\
\xi_{(5)\hat{\alpha}}^L &= (0, -\sin \alpha, \cos \alpha \sin \beta, 0, -\sin \alpha \cos \chi \cos \psi - \cos \alpha \cos \beta \sin \chi \cos \psi, \\
& \cos \alpha \sin \beta \cos \theta - \sin \alpha \sin \chi \cos \psi \sin \theta + \cos \alpha \cos \beta \cos \chi \cos \psi \sin \theta), \\
\xi_{(6)\hat{\alpha}}^L &= (1, 0, \cos \beta, 0, \sin \beta \sin \chi \cos \psi, \cos \beta \cos \theta - \sin \beta \cos \chi \cos \psi \sin \theta).
\end{aligned} \tag{A15}$$

The equations (A9) are given as

$$\begin{aligned}
f_{mnp} \Phi_p^a + \varepsilon^{abc} \Phi_m^b \Phi_n^c &= 0 \quad (a = 1, 2, 3, m = 1, \dots, 6, n = 4, 5, 6), \\
\partial_t \Phi_n^a - \varepsilon^{abc} A_i^b \Phi_n^c &= 0 \quad (a = 1, 2, 3, i = t, r, n = 4, 5, 6).
\end{aligned} \tag{A16}$$

This set of equations has two types of solutions; one is the “electric” type and the other is the “magnetic” one. The former type is given by

$$A_t^a = (0, 0, A_t), \quad A_r^a = (0, 0, A_r), \quad \text{and} \quad \Phi_m^a = 0, \tag{A17}$$

leading to the potential form as

$$\mathbf{A} = \tau_3 (A_t dt + A_r dr), \tag{A18}$$

Using a gauge freedom, we can set  $A_r = 0$ . The “electric” type of potential is now given by

$$\mathbf{A} = \tau_3 A(t, r) dt. \tag{A19}$$

While the latter type solution is given by

$$\begin{aligned}
A_t^a &= (0, 0, \dot{X}), & A_r^a &= (0, 0, X'), \\
\Phi_1^a &= (0, 0, \phi), & \Phi_2^a &= \pm(\phi \cos X, \phi \sin X, 0), & \Phi_3^a &= (\phi \sin X, -\phi \cos X, 0), \\
\Phi_4^a &= \pm(\sin X, -\cos X, 0), & \Phi_5^a &= -(\cos X, \sin X, 0), & \Phi_6^a &= (0, 0, \pm 1).
\end{aligned} \tag{A20}$$

We then obtain a general form of  $A_\mu^a$  as

$$\begin{aligned}
\mathbf{A} &= \tau_3 (\dot{X} dt + X' dr + \phi d\psi + \cos \theta d\varphi) + \cos \psi [(\tau_1 \sin X - \tau_2 \cos X) d\theta - (\tau_1 \cos X + \tau_2 \sin X) \sin \theta d\varphi] \\
&+ \phi \sin \psi [(\tau_1 \cos X + \tau_2 \sin X) d\theta + (\tau_1 \sin X - \tau_2 \cos X) \sin \theta d\varphi].
\end{aligned} \tag{A21}$$

$X$  is not a dynamical variable but it is regarded as a gauge variable. In fact, the field strength  $F_{\mu\nu}^a$  is given by

$$\begin{aligned}
\mathbf{F} &= \tau_3 \left[ \dot{\phi} dt \wedge d\psi + \phi' dr \wedge d\psi - (1 - \phi^2)(\sin \psi d\theta) \wedge (\sin \psi \sin \theta d\varphi) \right] \\
&+ (\tau_1 \cos X + \tau_2 \sin X) \left[ \dot{\phi} dt \wedge (\sin \psi d\theta) + \phi' dr \wedge (\sin \psi d\theta) + (1 - \phi^2) d\psi \wedge (\sin \psi \sin \theta d\varphi) \right] \\
&+ (\tau_1 \sin X - \tau_2 \cos X) \left[ \dot{\phi} dt \wedge (\sin \psi \sin \theta d\varphi) + \phi' dr \wedge (\sin \psi \sin \theta d\varphi) - (1 - \phi^2) d\psi \wedge (\sin \psi d\theta) \right].
\end{aligned} \tag{A22}$$

Rotating the  $\tau_1$ - $\tau_2$  plane of the interior space by  $-X$ , the variable  $X$  is eliminated. If we choose  $X = 0$ , we find

$$\begin{aligned}
\mathbf{A} &= \tau_3 (\phi d\psi + \cos \theta d\varphi) - \cos \psi [\tau_2 d\theta + \tau_1 \sin \theta d\varphi] + \phi \sin \psi [\tau_1 d\theta - \tau_2 \sin \theta d\varphi] \\
\mathbf{F} &= \tau_3 \left[ \dot{\phi} dt \wedge d\psi + \phi' dr \wedge d\psi - (1 - \phi^2)(\sin \psi d\theta) \wedge (\sin \psi \sin \theta d\varphi) \right] \\
&+ \tau_1 \left[ \dot{\phi} dt \wedge (\sin \psi d\theta) + \phi' dr \wedge (\sin \psi d\theta) + (1 - \phi^2) d\psi \wedge (\sin \psi \sin \theta d\varphi) \right] \\
&- \tau_2 \left[ \dot{\phi} dt \wedge (\sin \psi \sin \theta d\varphi) + \phi' dr \wedge (\sin \psi \sin \theta d\varphi) - (1 - \phi^2) d\psi \wedge (\sin \psi d\theta) \right].
\end{aligned} \tag{A23}$$

## APPENDIX B: NON-EXISTENCE OF FINITE MASS OBJECT ( $\Lambda \leq 0$ )

Here we show that there is no particle-like solution with finite mass if  $\Lambda \leq 0$  ( $\epsilon = 0$  or  $1$ ).

Introducing new variable

$$z = 2 \ln r \quad (\text{B1})$$

we rewrite the basic equations (3.1) and (3.3) with Eq. (3.2) as

$$\frac{d\mu}{dz} = 4f \left( \frac{dw}{dz} \right)^2 + (1 - w^2)^2 \quad (\text{B2})$$

$$f \frac{d^2 w}{dz^2} + \left[ e^{-z} \mu + \frac{\epsilon}{\ell^2} e^z - e^{-z} (1 - w^2)^2 \right] \frac{dw}{dz} + \frac{1}{2} w (1 - w^2) = 0 \quad (\text{B3})$$

with

$$f = 1 - e^{-z} \mu + \frac{\epsilon}{\ell^2} e^z, \quad (\text{B4})$$

where the function  $\delta$  is eliminated.

If we turn off gravity, that is, if we consider the Yang-Mills field equation in the Minkowski space, we have one basic equation

$$\frac{d^2 w}{dz^2} - e^{-z} (1 - w^2)^2 \frac{dw}{dz} + \frac{1}{2} w (1 - w^2) = 0. \quad (\text{B5})$$

This is easily integrated as

$$\frac{1}{2} \left( \frac{dw}{dz} \right)^2 - \frac{1}{8} (1 - w^2)^2 = E_0, \quad (\text{B6})$$

where  $E_0$  is an integration constant. Integrating this equation with the boundary condition  $w \rightarrow \pm 1$  as  $z \rightarrow -\infty$  ( $r \rightarrow 0$ ) and  $z \rightarrow \infty$  ( $r \rightarrow \infty$ ), which implies  $E_0 = 0$ , we obtain the solution for  $w$  as

$$w = \pm \tanh \frac{z}{2}. \quad (\text{B7})$$

This is exactly the same as the Yang-Mills instanton solution in 4-dimensional Euclidean spacetime [28]. If we regard

$$U(z) = -\frac{1}{8} (1 - w^2)^2 \quad (\text{B8})$$

as a potential, Eq. (B6) just denotes the energy conservation. The instanton corresponds to zero energy solution, in which  $w$  varies from  $\pm 1$  to  $\mp 1$  as  $z = -\infty \rightarrow \infty$ .

When we include the effect of gravity, whether we still have such a non-trivial structure or not? This is our question. In this appendix, we will show that there is no self-gravitating non-trivial solution with a finite mass energy. To discuss it, we introduce the energy function  $E$  by

$$E = \frac{1}{2} f \left( \frac{dw}{dz} \right)^2 - \frac{1}{8} (1 - w^2)^2. \quad (\text{B9})$$

The basic equations (B2) and (B3) are described as

$$\frac{dE}{dz} = -4e^{-z} \left( \frac{dw}{dz} \right)^2 \left[ E + \frac{\mu}{8} + \frac{\epsilon}{8\ell^2} e^{2z} \right] \quad (\text{B10})$$

$$\frac{d\mu}{dz} = 8E + 2(1 - w^2)^2. \quad (\text{B11})$$

Since we are interested in a particle-like solution, which must be regular at the origin, we can expand the functions  $\mu$  and  $w$  as

$$\begin{aligned} \mu &= \mu_1 e^z + \mu_2 e^{2z} + \mu_3 e^{3z} + \dots \\ w &= 1 + w_1 e^z + w_2 e^{2z} + w_3 e^{3z} + \dots, \end{aligned} \quad (\text{B12})$$

as  $z \rightarrow -\infty$  ( $r \rightarrow 0$ ). Inserting this form into Eqs. (B2) and (B3), we find the expansion coefficients as

$$\begin{aligned} \mu_1 &= 0 \quad \mu_2 = 4w_1^2 \quad \mu_3 = -\frac{4\epsilon}{\ell^2} w_1^2 + \frac{16}{3} w_1^3 (1 + w_1) \\ w_2 &= -\frac{2\epsilon}{3\ell^2} w_1 + \frac{w_1^2}{6} (3 + 8w_1) \\ w_3 &= \frac{\epsilon^2}{2\ell^4} w_1 - \frac{\epsilon}{8\ell^2} w_1^2 (5 + 24w_1) \\ &\quad + \frac{w_1^3}{4} (1 + 8w_1)(1 + 2w_1) \end{aligned} \quad (\text{B13})$$

where  $w_1$  is a free parameter.

Putting those relations in Eqs. (B9) and (B12), we find

$$E = -\frac{1}{6} w_1^2 e^{3z} \left( \frac{\epsilon}{\ell^2} + 4w_1^2 \right) \quad (\text{B14})$$

$$\mu = 4w_1^2 e^{2z}. \quad (\text{B15})$$

For  $\epsilon = 0$  or  $1$ ,  $E \rightarrow -0$  and  $\mu \rightarrow +0$  as  $z \rightarrow -\infty$ . The r.h.s. of Eq. (B11) is positive definite because

$$8E + 2(1 - w^2)^2 = 4f \left( \frac{dw}{dz} \right)^2 + (1 - w^2)^2, \quad (\text{B16})$$

and  $f > 0$  should be imposed for a particle-like solution. Hence, the mass function  $\mu$  is also positive definite.

Next, we analyze the behavior of the solution near infinity ( $z \rightarrow \infty$ ). If the mass function does not diverge, we can expand  $\mu$  and  $w$  as

$$\begin{aligned} \mu &= \mathcal{M}_0 + \mathcal{M}_1 e^{-z} + \mathcal{M}_2 e^{-2z} + \mathcal{M}_3 e^{-3z} + \dots \\ w &= -1 + \mathcal{W}_1 e^{-z} + \mathcal{W}_2 e^{-2z} + \mathcal{W}_3 e^{-3z} + \dots, \end{aligned} \quad (\text{B17})$$

as  $z \rightarrow \infty$ .

From the basic equations, we find the relations between the expansion coefficients as

$$\begin{aligned} \mathcal{M}_1 &= 0, \quad \mathcal{M}_2 = -4\mathcal{W}_1^2 \\ \mathcal{M}_3 &= \frac{4}{3} \mathcal{W}_1^2 (4\mathcal{W}_1 - 3\mathcal{M}_0) \\ \mathcal{W}_2 &= \frac{2}{3} \mathcal{M}_0 \mathcal{W}_1 - \frac{1}{2} \mathcal{W}_1^2 \\ \mathcal{W}_3 &= \frac{\mathcal{W}_1}{8} (4\mathcal{M}_0^2 - 5\mathcal{M}_0 \mathcal{W}_1 + 2\mathcal{W}_1^2) \end{aligned} \quad (\text{B18})$$

for  $\epsilon = 0$ , and

$$\begin{aligned}\mathcal{M}_1 &= -4\frac{\mathcal{W}_1^2}{\ell^2} \quad \mathcal{M}_2 = -\frac{8}{\ell^2}\mathcal{W}_1\mathcal{W}_2 \\ \mathcal{M}_3 &= -\frac{2}{3}\mathcal{W}_1^2(4\mathcal{M}_0 - 3\mathcal{W}_1) \quad \mathcal{W}_2 = 0 \\ \mathcal{W}_3 &= \frac{\ell^2}{12}\mathcal{W}_1(4\mathcal{M}_0 - 3\mathcal{W}_1)\end{aligned}\quad (\text{B19})$$

for  $\epsilon = 1$ . Here,  $\mathcal{M}_0$  and  $\mathcal{W}_1$  are free parameters.

Using those relations, the energy function  $E$  near infinity is evaluated as

$$E = \frac{1}{6}e^{-3z}\mathcal{W}_1^2 + \dots \rightarrow +0 \quad (\text{B20})$$

for  $\epsilon = 0$

$$E = \frac{1}{2\ell^2}e^{-z}\mathcal{W}_1^2 + \dots \rightarrow +0 \quad (\text{B21})$$

for  $\epsilon = 1$ .

Since  $E \rightarrow -0$  near the origin while  $E \rightarrow +0$  at infinity, if the solution is regular everywhere,  $E$  must vanishes at some finite point ( $z_0$ ) and  $dE/dz \geq 0$  there. On the other hand, Eq. (B10) yields  $dE/dz \leq 0$  since  $E(z_0) = 0$  and  $\mu(z_0) > 0$ . As a result, we have  $dE/dz(z_0) = 0$ . Using Eq. (B10), we then find  $dw/dz(z_0) = 0$ .  $E(z_0) = 0$  with this equation implies  $w(z_0) = \pm 1$ . Solving the basic equations (B2) and (B3) with the above initial values at  $z_0$  ( $w(z_0) = \pm 1, dw/dz(z_0) = 0, \mu(z_0) = \text{positive and finite}$ ), we find a trivial solution ( $w(z) = \pm 1, \mu(z) = \text{a positive constant}$ ). We conclude that there is no non-trivial particle-like solution with a finite mass for  $\Lambda \leq 0$ .

### APPENDIX C: ASYMPTOTIC SOLUTION ( $\Lambda \leq 0$ )

We present the asymptotic solution of the present system with  $\Lambda \leq 0$ . As we proved for a particle-like solution in the previous appendix and numerically solved for more generic case, the mass function  $\mu$  seems to diverge. Here we solve the basic equations with some ansatz and find the analytic solution in the asymptotically far region.

First, we consider the case of  $\epsilon = 0$ . As our ansatz, we adopt

$$\mu \approx \mathcal{M}_L z, \quad (\text{C1})$$

which is suggested from numerical solutions and also confirmed from the following result. The basic equation for the Yang-Mills field is now written as

$$\begin{aligned}\frac{d^2 w}{dz^2} + e^{-z}\mu \frac{dw}{dz} + \frac{1}{2}w(1-w^2) \\ \approx \frac{d^2 w}{dz^2} + \frac{1}{2}w(1-w^2) = 0,\end{aligned}\quad (\text{C2})$$

as  $z \rightarrow \infty$ . We can integrate this equation as

$$\frac{1}{2} \left( \frac{dw}{dz} \right)^2 - \frac{1}{8} (1-w^2)^2 = E_0, \quad (\text{C3})$$

where  $E_0$  is an integration constant and denotes the asymptotic value of the energy.  $E_0$  must be negative, otherwise  $w$  diverges as  $z \rightarrow \infty$ .

Rewriting Eq. (C3), we find

$$\begin{aligned}\frac{dw}{dz} &= \pm \frac{1}{2} \sqrt{8E_0 + (1-w^2)^2} \\ &= \pm \frac{1}{2} \sqrt{(w_-^2 - w^2)^2 (w_+^2 - w^2)^2},\end{aligned}\quad (\text{C4})$$

where

$$w_{\pm} = \sqrt{1 \pm 2\sqrt{-2E_0}}, \quad (\text{C5})$$

which is integrated as

$$w = \pm w_- \text{sn} \left( \frac{w_+}{2} z, k \right), \quad (\text{C6})$$

where  $k = w_-/w_+$ .  $w$  is oscillating in a potential  $U(z) = -\frac{1}{8}(1-w^2)^2$  with a negative energy  $E_0$ .

In order to check out ansatz, we also solve the mass function with the above solution of  $w$ . The mass function  $\mu$  is obtained by integration of Eq. (B11), that is

$$\begin{aligned}\mu &= \int dz (8E_0 + 2(1-w^2)^2) \\ &= \mu_0 z + 4w_+ w_-^2 \int^{w_+ z/2} dx [(1-k^2)\text{cn}^2(x, k) \\ &\quad + k^2 \text{cn}^4(x, k)],\end{aligned}\quad (\text{C7})$$

where  $\mu_0 = 8E_0 + 2(1-w_-^2)^2 = -8E_0 = (1-w_-^2)^2$ . The integration of the functions  $\text{cn}^2$  and  $\text{cn}^4$  is evaluated by the elliptic functions as:

$$\begin{aligned}\int dx \text{cn}^2(x, k) &= \frac{1}{k^2} [-(1-k^2)x \\ &\quad + \frac{E(\sin^{-1}(\text{sn}(x, k)), k) \times (1-k^2 \text{sn}^2(x, k))}{\text{dn}^2(x, k)}] \\ \int dx \text{cn}^4(x, k) &= \frac{1}{3k^2} [(1-k^2)x \\ &\quad + \text{cn}(x, k)\text{dn}(x, k)\text{sn}(x, k)] + \left[ \frac{2(2k^2-1)}{3k^4} (-(1-k^2)x \right. \\ &\quad \left. + \frac{E(\sin^{-1}(\text{sn}(x, k)), k) \times (1-k^2 \text{sn}^2(x, k))}{\text{dn}^2(x, k)} \right)\end{aligned}\quad (\text{C8})$$

There functions increase with oscillations as  $z \rightarrow \infty$ . When we take an average of those functions over the period of oscillation, the averaged values are linearly increasing as our ansatz (C1).

Dividing those functions into two parts (linear functions and oscillating functions), we find

$$\mu = \mathcal{M}_L z + \frac{8\sqrt{2}k^2}{(1+k^2)^{3/2}} D\mu \left( \frac{w_+}{2} z, k \right), \quad (\text{C9})$$

where

$$\mathcal{M}_L = \frac{1}{(1+k^2)^2} [(1-k^2)^2 + 8k^2\{(1-k^2)C_2 + k^2C_4\}] \quad (\text{C10})$$

$$D\mu(x, k) = (1-k^2) \int dx [\text{cn}^2(x, k) - C_2] + k^2 \int dx [\text{cn}^4(x, k) - C_4] \quad (\text{C11})$$

$$C_2 = \frac{1}{K(k)} \int_0^{K(k)} dx \text{cn}^2(x, k) \\ C_4 = \frac{1}{K(k)} \int_0^{K(k)} dx \text{cn}^4(x, k) \quad (\text{C12})$$

The energy  $E_0$  and the amplitude  $w_-$  are found to be

$$E_0 = -\frac{1}{8} \left( \frac{1-k^2}{1+k^2} \right)^2 \\ w_- = \frac{\sqrt{2}k}{\sqrt{1+k^2}}. \quad (\text{C13})$$

with  $k$  ( $0 \leq k \leq 1$ ).

The asymptotic solution is then given as

$$w = \pm w_- \text{sn} \left( \frac{w_+}{2} z, k \right) \quad (\text{C14})$$

$$\mu = \mathcal{M}_L z + \frac{8\sqrt{2}k^2}{(1+k^2)^{3/2}} D\mu \left( \frac{w_+}{2} z, k \right), \quad (\text{C15})$$

$D\mu$  is a periodic function with a constant period, which in  $z$ -coordinate is

$$\Delta z = \frac{8}{w_+} K(k) = \frac{4\pi}{w_+} F \left( \frac{1}{2}, \frac{1}{2}, 1, k^2 \right), \quad (\text{C16})$$

where

$$w_+ = \frac{\sqrt{2}}{\sqrt{1+k^2}} \quad (\text{C17})$$

Note that if we take a limit of  $k \rightarrow 1$ ,  $E_0 \rightarrow -0$ , we recover the instanton solution, that is,  $w$  is oscillating between  $\pm 1$ . The width of one instanton ( $w \sim \pm 1 \rightarrow \mp 1$ ) is given by  $\Delta z/2$ , which diverges in this limit. However this oscillation is repeated infinitely when a gravitational effect is included, that is infinite number of instantons appear in the present system. This is why the mass function diverges.

For the case of  $\epsilon = 1$  ( $\Lambda < 0$ ), since  $f \rightarrow \ell^{-2}$ , the equation for  $w$  is now

$$\frac{d^2 w}{dz^2} + \frac{dw}{dz} + \frac{\ell^2}{2} e^{-z} w (1-w^2) \\ \approx \frac{d^2 w}{dz^2} + \frac{dw}{dz} = 0. \quad (\text{C18})$$

Integrating this equation, we obtain that

$$\left| \frac{dw}{dz} \right| \propto e^{-z}, \quad (\text{C19})$$

which gives the asymptotic behaviors of  $w$  as

$$w = \mathcal{W}_0 + \mathcal{W}_1 e^{-z} + \dots \quad (\text{C20})$$

The equation for  $\mu$  is

$$\frac{d\mu}{dz} = \frac{4}{\ell^2} \left( \frac{dw}{dz} \right)^2 + (1-w^2)^2 \\ \approx (1-\mathcal{W}_0^2)^2 + 4\mathcal{W}_1 \left[ \frac{\mathcal{W}_1}{\ell^2} - \mathcal{W}_0(1-\mathcal{W}_0^2) \right] e^{-z} + \dots \quad (\text{C21})$$

and then  $\mu$  is given by

$$\mu = \mathcal{M}_L z + \mathcal{M}_0 + \mathcal{M}_1 e^{-z} + \dots, \quad (\text{C22})$$

where

$$\mathcal{M}_L = (1-\mathcal{W}_0^2)^2 \\ \mathcal{M}_1 = -4\mathcal{W}_1 \left[ \frac{1}{\ell^2} \mathcal{W}_1 - \mathcal{W}_0(1-\mathcal{W}_0^2) \right] \quad (\text{C23})$$

The energy function  $E$  is evaluated as

$$E = \frac{1}{2\ell^2} e^{-z} \left( \frac{dw}{dz} \right)^2 - \frac{1}{8} (1-w^2)^2 \\ \rightarrow -\frac{1}{8} (1-\mathcal{W}_0^2)^2 (= E_0). \quad (\text{C24})$$

Since the damping rate of the energy is given by

$$\frac{dE}{dz} = -4e^{-z} \left( \frac{dw}{dz} \right) \left[ E + \frac{\mu}{8} + \frac{1}{8\ell^2} e^{2z} \right] \\ \rightarrow -\frac{1}{2\ell^2} \mathcal{W}_1^2 e^{-z}, \quad (\text{C25})$$

the energy damping ceases very soon. As a result, the energy of the system approaches some finite value ( $E_0$ ). This is because the potential term drops exponentially while the adiabatic damping term remains. The solution does not oscillate because the potential term becomes ineffective quickly.

- 
- [1] P. Horava and E. Witten, Nucl. Phys. **460**, 506 (1996)
  - [2] J. Dai, R. G. Leigh and J. Polchinski, Mod. Phys. Lett. A **4**, 2073 (1989)
  - [3] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **565**, 269 (2000); N. Kaloper, Phys. Rev. D **60**, 123506 (1999); C. Csaki, M. Graesser C. Kolda and J. Terning,



- Phys. Lett. B **462**, 34 (1999); T. Nihei, Phys. Lett. B **465**, 81 (1999) P. Kanti, I. I. Kogan, K. A. Olive and M. Prosser, Phys. Lett. B **468**, 31 (1999); J. M. Cline, C. Grojean and G. Servant, Phys. Rev. Lett. **83**, 4245 (1999); P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B **477**, 285 (2000); S. Mukohyama, T. Shiromizu and K. Maeda, Phys. Rev. D **62**, 024028 (2000)
- [4] P. Kraus J. High Energy Phys. **12**, 011 (1999)
  - [5] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998)
  - [6] A. Strominger, J. High Energy Phys. **0110**, 034 (2001)
  - [7] C. Csaki, J. Erlich and C. Grojean, Nucl. Phys. B **604**, 312 (2001)
  - [8] R. Bartnik and J. McKinnon, Phys. Rev. Lett. **61**, 141 (1988)
  - [9] P. Bizon, Phys. Rev. Lett. **64**, 2844 (1990)
  - [10] M. S. Volkov, N. Straumann, G. Lavrelashvili, M. Heusler and O. Brodbeck, Phys. Rev. D **54**, 7243 (1996)
  - [11] T. Torii, K. Maeda and T. Tachizawa, Phys. Rev. D **52**, R4272 (1995)
  - [12] J. Bjoraker and Y. Hosotani, Phys. Rev. Lett. **84**, 1853 (2000)
  - [13] O. Brodbeck, M. Heusler, G. Lavrelashvili, N. Straumann and M. S. Volkov, Phys. Rev. D **54**, 7338 (1996)
  - [14] E. Winstanley, Class. Quantum Grav. **16**, 1963 (1999)
  - [15] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999)
  - [16] A. Lukas, B. A. Ovrut, K. S. Stelle and D. Waldram Phys. Rev. D **59**, 086001 (1999)
  - [17] M. Kubo, C. S. Lim and H. Yamashita, hep-ph/0111327
  - [18] R. Arnowitt, S. Deser and C. W. Misner, *Gravitation; An Introduction to Current Research*, ed. L. Witten (New York: Wiley) (1962)
  - [19] L. Abbott and S. Deser, Nucl. Phys. B **195**, 76 (1982)
  - [20] K. Nakao, T. Shiromizu, K. Maeda, Class. Quantum Grav. **11**, 2059 (1994)
  - [21] A. Ashtekar and A. Magnon, Class. Quantum Grav. **1**, L39 (1984)
  - [22] A. Ashtekar and S. Das, Class. Quantum Grav. **17**, L17 (2000)
  - [23] U. Nucamendi and D. Sudarsky, Class. Quantum Grav. **14**, 1309 (1997)
  - [24] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975)
  - [25] E. Witten, Phys. Rev. Lett. **38**, 121 (1977)
  - [26] P. Forgács and N. S. Manton, Commun. Math. Phys. **72**, 15 (1980)
  - [27] P. G. Bergmann and E. J. Flaherty, Jr., J. Math. Phys. **19**, 212 (1978)
  - [28] P. Bizon and Z. Tabor, Phys. Rev. D **64**, 121701 (2001)